Algorithms and Data Structures

CS-CO-412

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Algorithmic Strategies

Lecture 15

Topic Overview

- **Brute-force**
- **Divide-and-conquer**
- **Greedy algorithms**
- **Dynamic Programming**
- Combinatorial Search & Backtracking
- Branch-and-bound

Brute Force

- Brute force is a straightforward approach to solve a problem based on a simple formulation of problem
- Often without any deep analysis of the problem
- Perhaps the easiest approach to apply and is useful for solving small-size instances of a problem
- May result in naïve solutions with poor performance
- Some examples of brute force algorithms are:
  - Computing a^n ($a > 0$, n a non-negative integer) by repetitive multiplication: $a \times a \times \ldots \times a$
  - Computing $n!$
  - Sequential search
  - Selection sort, Bubble sort

Brute Force

- Maximum subarray problem / Grenander's Problem
  - Given a sequence of integers $i_1, i_2, \ldots, i_n$, find the sub-sequence with the maximum sum
    - If all numbers are negative the result is 0
  - Examples:

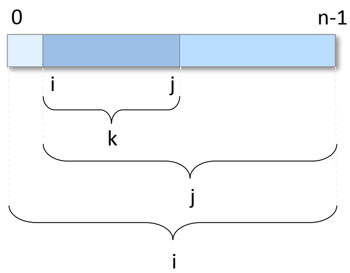
    -2, 11, -4, 13, -4, 2 gives the solution 20

    1, -3, 4, -2, -1, 6 gives the solution 7

Brute Force

- Maximum subarray problem: brute force solution O(n^3)

```
int grenanderBF(int a[], int n) {
   int maxSum = 0;
   for (int i = 0; i < n; i++) {
      for (int j = i; j < n; j++) {
         int thisSum = 0;
         for (int k = i; k <= j; k++) {
            thisSum += a[ k ];
         }
         if (thisSum > maxSum) {
            maxSum = thisSum;
         }
      }
   }
   return maxSum;
}
```



Brute Force

- Maximum subarray problem
  - Divide and Conquer algorithm O(n log n)
  - Kadane's algorithms O(n) … dynamic programming

Divide and Conquer

- Divide-and conquer (D&Q)
  - Given an instance of the problem
  - Divide this into smaller sub-instances (often two)
  - Independently solve each of the sub-instances
  - Combine the sub-instance solutions to yield a solution for the original instance
- With the D&Q method, the size of the problem instance is reduced by a factor (e.g. half the input size)

Divide and Conquer

- Often yield a recursive formulation
- Examples of D&Q algorithms
  - Quicksort algorithm
  - Mergesort algorithm
  - Fast Fourier Transform

Divide and Conquer

Mergesort

UNSORTEDSEQUENCE

UNSORTED SEQUENCE

UNSO RTED SEQU ENCE

UN SO RT ED SE QU EN CE

NU OS RT DE ES QU EN CE

NOSU DERT EQSU CEEN

DENORSTU CEEENQSU

CDEEEENNOQRSSTUU

Divide and Conquer

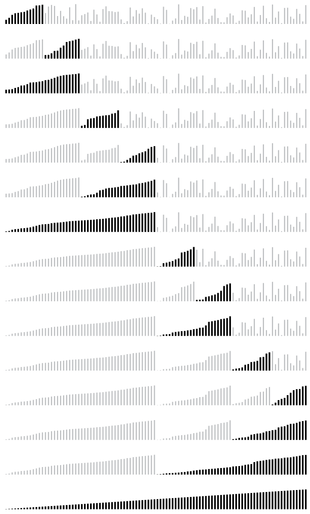
```
void mergesort(Item a[], int l, int r) {
    if (r-l <= 1) {
        return;
    } else {
        int m = (r + l) / 2;
        mergesort(a, l, m);
        mergesort(a, m+1, r);
        merge(a, l, m, r);
    }
}

void mergesort(Item a[], int size) {
    mergesort(a, 0, size-1);
}
```

Already sorted?

Divide the list into two equal parts

Sort the two halves recursively

Merge the sorted halves into a sorted whole



Divide and Conquer

```
int grenanderDQ(int a[], int l, int h) {
    if (l > h) return 0;
    if (l = h) return max(0, a[l]);
    int m = (l + h) / 2;
    int sum = 0;
    int maxLeft = 0;
    for (int i = m; i >= l; i--) {
        sum += a[i];
        maxLeft = max(maxLeft, sum);
    }
    sum = 0;
    int maxRight = 0;
    for (int i = m + 1; i <= h; i++) {
        sum += a[i];
        maxRight = max(maxRight, sum);
    }
    int maxL = grenanderDQ(a, l, m);
    int maxR = grenanderDQ(a, m+1, h);
    int maxC = maxLeft + maxRight;
    return max(maxC, max(maxL, maxR));
}
```

Divide the problem

Solve the sub-problem

Solve the sub-problem

Solve the sub-problems

Combine the solutions

Divide and Conquer

```
// Generic Divide and Conquer Algorithm

divideAndConquer(Problem p) {
   if (p is simple or small enough) {
      return simpleAlgorithm(p);
   } else {
      divide p in smaller instances p1, p2, ..., pn
      Solution solutions[n];
      for (int i = 0; i < n; i++) {
         solutions[i] = divideAndConquer(pi);
      }
      return combine(solutions);
   }
}
```

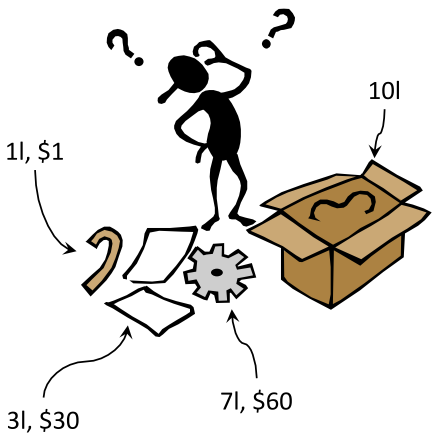

Greedy Algorithms

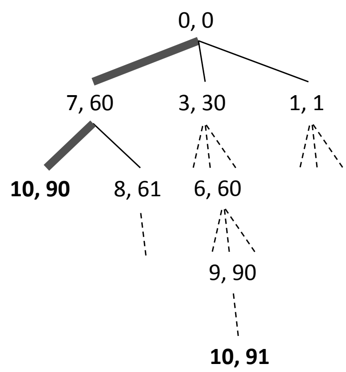
- Try to find solutions to problems step-by-step
  - A partial solution is incrementally expanded towards a complete solution
  - In each step, there are several ways to expand the partial solution:
  - The best alternative for the moment is chosen, the others are discarded
- At each step the choice must be **locally optimal** – this is the central point of this technique

Greedy Algorithms

- Examples of problems that can be solved using a greedy algorithm:
  - Finding the minimum spanning tree of a graph (Prim's algorithm)
  - Finding the shortest distance in a graph (Dijkstra's algorithm)
  - Using Huffman trees for optimal encoding of information
  - The Knapsack problem

Greedy Algorithms





Dynamic Programming

- Dynamic programming is similar to D&Q
  - Divides the original problem into smaller sub-problems
- Sometimes it is hard to know beforehand which sub-problems are needed to be solved in order to solve the original problem
- Dynamic programming solves a large number of sub-problems and uses some of the sub-solutions to form a solution to the original problem

Dynamic Programming

- In an optimal sequence of choices, actions or decisions each sub-sequence must also be optimal:
  - An optimal solution to a problem is a combination of optimal solutions to some of its sub-problems
  - Not all optimization problems adhere to this principle

Dynamic Programming

- One disadvantage of using D&Q is that the process of recursively solving separate sub-instances can result in **the same computations being performed repeatedly**
- The idea behind dynamic programming is to avoid calculating the same quantity twice, usually by **maintaining a table of sub-instance results**

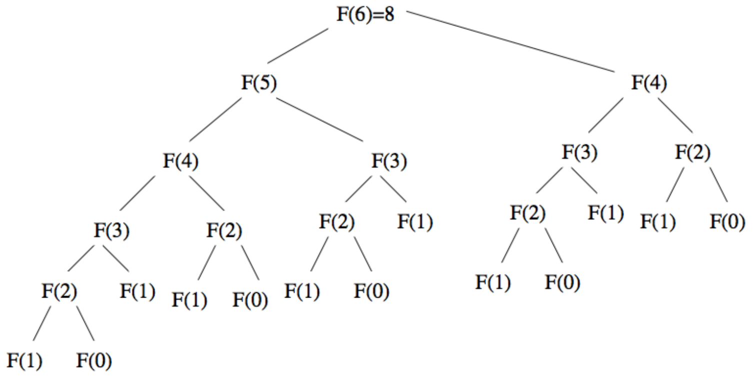
Dynamic Programming

- The same sub-problems may reappear
- To avoid solving the same sub-problem more than once, sub-results are saved in a data structure that is updated dynamically
- Sometimes the result structure (or parts of it) may be computed beforehand

Dynamic Programming

```
/* fibonacci by recursion O(1.618^n) time complexity */

long fib_r(int n) {
   if (n == 0)
      return(0);
   else
      if (n == 1)
         return(1);
      else
         return(fib_r(n-1) + fib_r(n-2));
}


fib_r(4)  → fib(3) + fib(2)
          → fib(2) + fib(1) + fib(2)
          → fib(1) + fib(0) + fib(1) + fib(2)
          → fib(1) + fib(0) + fib(1) + fib(1) + fib(0)
```

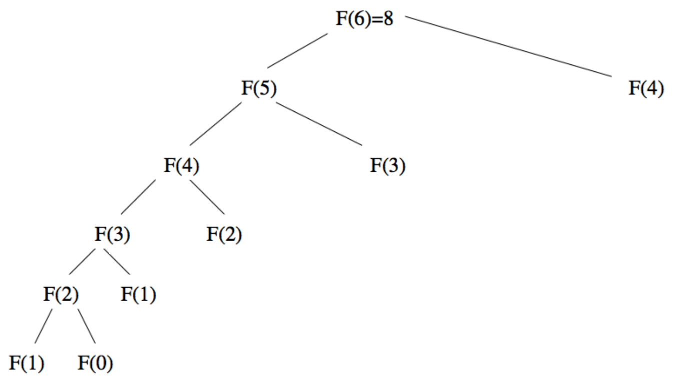

Dynamic Programming



Dynamic Programming

```
#define MAXN 45     /* largest interesting n                     */
#define UNKNOWN -1  /* contents denote an empty cell             */
long f[MAXN+1];     /* array for caching computed fib values     */

/* fibonacci by caching: O(n) storage & O(n) time                */

long fib_c(int n) {
   if (f[n] == UNKNOWN)
      f[n] = fib_c(n-1) + fib_c(n-2);
   return(f[n]);
}

long fib_c_driver(int n) {
   int i; /* counter */

   f[0] = 0;
   f[1] = 1;
   for (i=2; i<=n; i++)
      f[i] = UNKNOWN;
   return(fib_c(n));
}
```


Dynamic Programming



Dynamic Programming

```
/* fibonacci by dynamic programming: cache & no recursion           */
/* NB: need correct order of evaluation in the recurrence relation  */
/* O(1) storage & O(n) time                                         */


long fib_dp(int n) {
   int i; /* counter */
   long f[MAXN+1]; /* array to cache computed fib values */

   f[0] = 0;
   f[1] = 1;

   for (i=2; i<=n; i++)
      f[i] = f[i-1]+f[i-2];

   return(f[n]);
}
```


Dynamic Programming

```
/* fibonacci by dynamic programming: minimal cache & no recursion       */
/* O(1) storage & O(n) time                                             */

long fib_ultimate(int n) {

   int i;                  /* counter */
   long back2=0, back1=1;  /* last two values of f[n] */
   long next;              /* placeholder for sum */

   if (n == 0) return (0);

   for (i=2; i<n; i++) {
      next = back1+back2;
      back2 = back1;
      back1 = next;
   }
   return(back1+back2);
}
```

Dynamic Programming

```
int grenanderDP(int a[], int n) {
  int table[n+1];
  table[0] = 0;
  for (int k = 1; k <= n; k++)
    table[k] = table[k-1] + a[k-1];
  int maxSoFar = 0;
  for (int i = 1; i <= n; i++)
    for (int j = i; j <= n; j++) {
      thisSum = table[ j ] - table[i-1];
      if (thisSum > maxSoFar)
        maxSoFar = thisSum;
    }
  return maxSum;
}
```


Dynamic Programming

- There are three steps involved in solving a problem by dynamic programming:

  1. Formulate the answer as a recurrence relation or recursive algorithm

  2. Show that the number of different parameter values taken on by your recurrence is bounded by a (hopefully small) polynomial

  3. Specify an order of evaluation for the recurrence so the partial results you need are always available when you need them

