Robotics: Principles and Practice

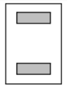
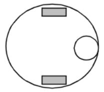
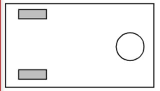
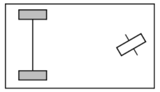
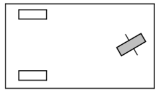
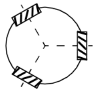
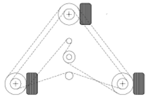
Module 3: Mobile Robots

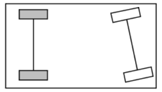
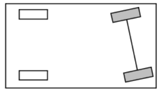
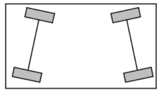
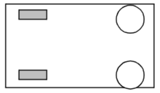
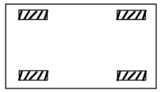
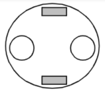
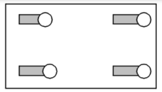
Lecture 5: Kinematics of a two-wheel differential drive robot

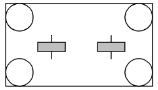
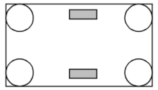
David Vernon
Carnegie Mellon University Africa

www.vernon.eu

Wheeled Locomotion

# of wheels	Arrangement	Description	Typical examples
2		One steering wheel in the front, one traction wheel in the rear	Bicycle, motorcycle
		Two-wheel differential drive with the center of mass (COM) below the axle	Cye personal robot
3		Two-wheel centered differential drive with a third point of contact	Nomad Scout, smartRob EPFL
		Two independently driven wheels in the rear/front, 1 unpowered omnidirectional wheel in the front/rear	Many indoor robots, including the EPFL robots Pygmalion and Alice
		Two connected traction wheels (differential) in rear, 1 steered free wheel in front	Piaggio minitrucks
		Two free wheels in rear, 1 steered traction wheel in front	Neptune (Carnegie Mellon University), Hero-1
		Three motorized Swedish or spherical wheels arranged in a triangle; omnidirectional movement is possible	Stanford wheel Tribolo EPFL, Palm Pilot Robot Kit (CMU)
		Three synchronously motorized and steered wheels; the orientation is not controllable	"Synchro drive" Denning MRV-2, Georgia Institute of Technology, I-Robot B24, Nomad 200

# of wheels	Arrangement	Description	Typical examples
4		Two motorized wheels in the rear, 2 steered wheels in the front; steering has to be different for the 2 wheels to avoid slipping/skidding.	Car with rear-wheel drive
		Two motorized and steered wheels in the front, 2 free wheels in the rear; steering has to be different for the 2 wheels to avoid slipping/skidding.	Car with front-wheel drive
		Four steered and motorized wheels	Four-wheel drive, four-wheel steering Hyperion (CMU)
		Two traction wheels (differential) in rear/front, 2 omnidirectional wheels in the front/rear	Charlie (DMT-EPFL)
		Four omnidirectional wheels	Carnegie Mellon Uranus
		Two-wheel differential drive with 2 additional points of contact	EPFL Khepera, Hyperbot Chip
		Four motorized and steered castor wheels	Nomad XR4000

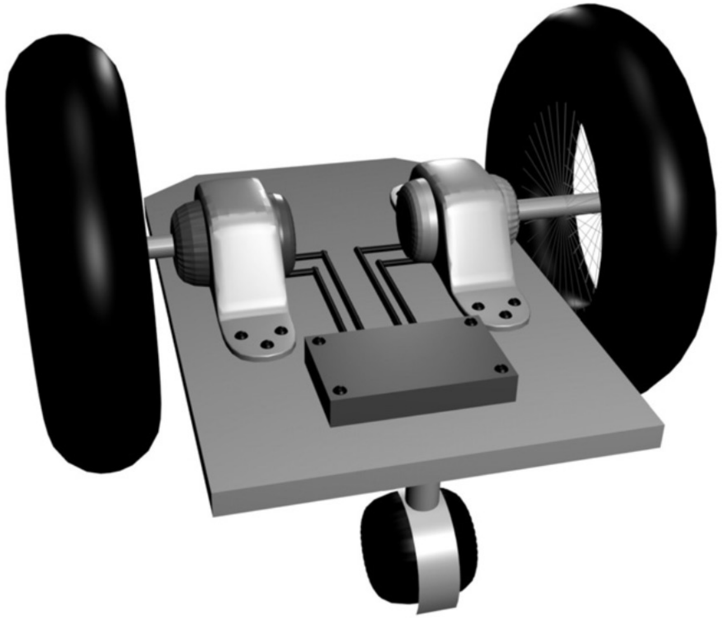
# of wheels	Arrangement	Description	Typical examples
6		Two motorized and steered wheels aligned in center, 1 omnidirectional wheel at each corner	First
		Two traction wheels (differential) in center, 1 omnidirectional wheel at each corner	Terregator (Carnegie Mellon University)

Icons for the each wheel type are as follows:	
	unpowered omnidirectional wheel (spherical, castor, Swedish);
	motorized Swedish wheel (Stanford wheel);
	unpowered standard wheel;
	motorized standard wheel;
	motorized and steered castor wheel;
	steered standard wheel;
	connected wheels.

We will study two-wheel differential drive locomotion

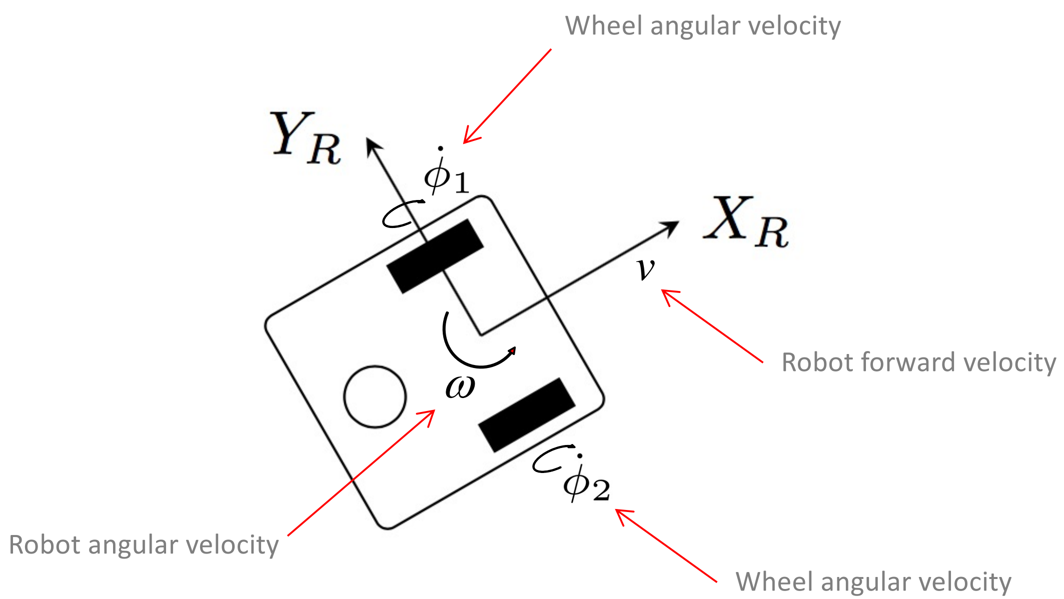
Source: R. Siegwart and I. R. Nourbakhsh, *Introduction to Autonomous Mobile Robots*, MIT Press, 2004

Wheeled Locomotion



Source: M. Mataric, The Robotics Primer, MIT Press, 2007

Wheeled Locomotion



Odometry-based Position Estimation

We need to know how to compute these three transformations

- Measure the angular velocities of the left and right wheels — $\dot{\phi}_1 \quad \dot{\phi}_2$
- Compute the instantaneous velocities in the robot frame of reference — v, ω
- Compute the displacement and change in orientation (in a given time interval) in the robot frame of reference R — $\Delta\xi_R = (\Delta x, \Delta y, \Delta\theta)$
- Compute the displacement and change in orientation (in a given time interval) in the global frame of reference (inertial frame of reference I) — $\Delta\xi_I = (\Delta x_I, \Delta y_I, \Delta\theta)$
- Update the position of the robot with respect to its previous position
- Repeat update (e.g. every 100 ms)

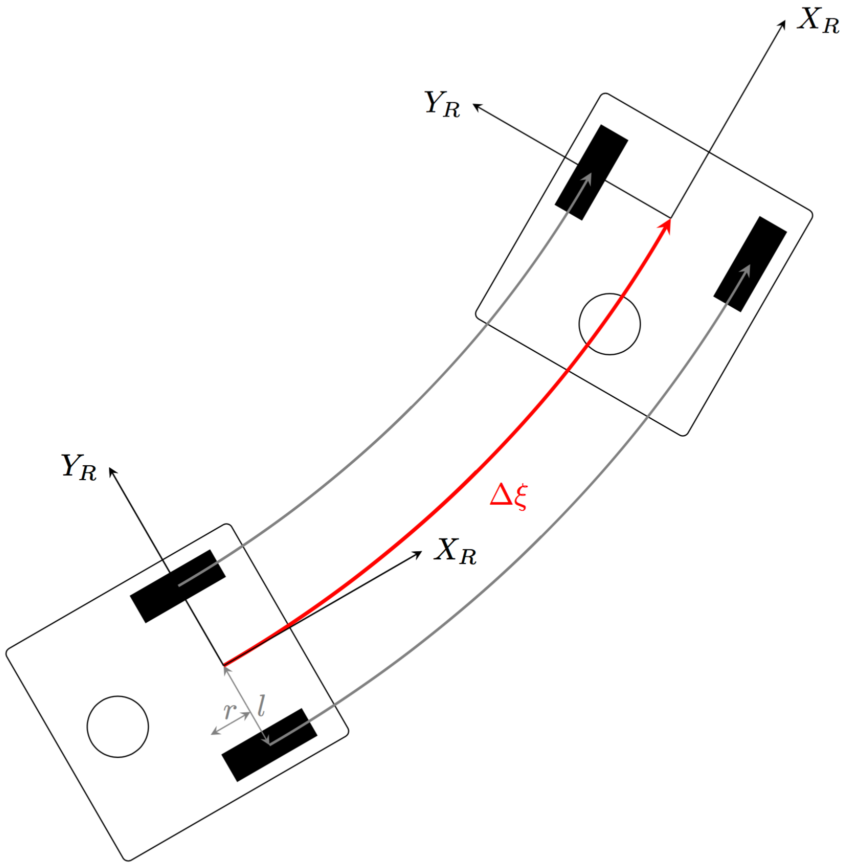
Instantaneous velocities in the robot frame of reference

We use the robot's forward kinematics

- Given the angular velocity of the wheels and the geometry of the robot

$\dot{\phi}_1$	right wheel angular velocity
$\dot{\phi}_2$	left wheel angular velocity
r	wheel radius
l	distance of the wheel from the origin of the robot frame of reference

- Determine the change in pose of the robot $\Delta\xi = (\Delta x, \Delta y, \Delta\theta)$



Odometry-based Position Estimation

Overall approach

- Measure the angular velocities of the left and right wheels $\dot{\phi}_1 \quad \dot{\phi}_2$
- Compute the instantaneous velocities in the robot frame of reference $\downarrow$ v, ω
- Compute the displacement and change in orientation (in a given time interval) in the robot frame of reference R
- Compute the displacement and change in orientation (in a given time interval) in the global frame of reference (inertial frame of reference I)
- Update the position of the robot with respect to its previous position
- Repeat update (e.g. every 100 ms)

Instantaneous velocities in the robot frame of reference

The motion of the robot in the local robot frame of reference R due to the rotation of the wheels is given by:

Also see Siegwart et al. (2004, 2011)

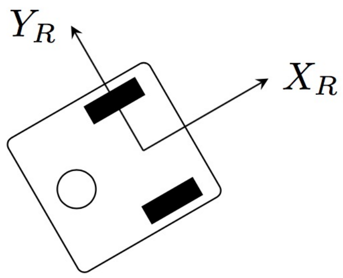
Instantaneous velocity v in the X_R direction

$$\dot{\xi}_R = \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l} \end{bmatrix}$$

Instantaneous velocity in the Y_R direction

Angular velocity ω about the vertical

Rate of change of pose in the robot frame of reference



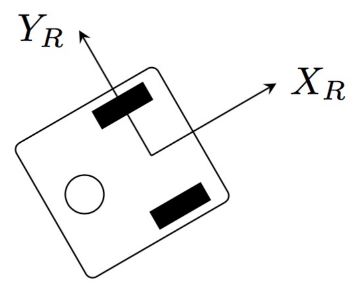
Instantaneous velocities in the robot frame of reference

The motion of the robot in the local robot frame of reference R due to the rotation of the wheels is given by:

Also see Siegwart and Nourkbakhsh. (2004, p. 52, 2011)

Instantaneous velocity v in the X_R direction

$$\dot{\xi}_R = \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l} \end{bmatrix}$$



Contribution to forward velocity by the right wheel

If the left wheel is stopped and the right wheel spins, then the instantaneous velocity of the origin of the robot's frame of reference – written {R} – will be half the angular velocity times the radius. Why half? Because the origin is half way between the right moving wheel and the left stopped wheel.

Contribution to forward velocity by the left wheel

If the right wheel is stopped and the left wheel spins, then the instantaneous velocity of the origin of the robot's frame of reference – written {R} – will be half the angular velocity times the radius. Why half? Because the origin is half way between the left moving wheel and the right stopped wheel.

Instantaneous velocities in the robot frame of reference

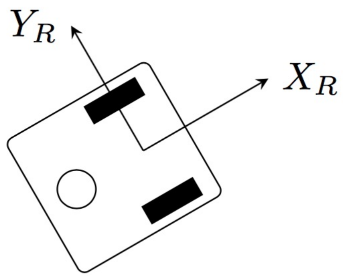
The motion of the robot in the local robot frame of reference R due to the rotation of the wheels is given by:

Also see Siegwart et al. (2004, 2011)

$$\dot{\xi}_R = \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l} \end{bmatrix}$$

Instantaneous velocity in the Y_R direction

Must be zero because the robot cannot move along the line joining the two wheels, i.e. in the YR direction



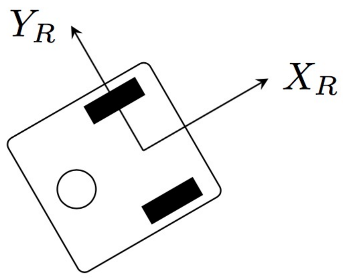
Instantaneous velocities in the robot frame of reference

The motion of the robot in the local robot frame of reference R due to the rotation of the wheels is given by:

Also see Siegwart et al. (2004, 2011)

$$\dot{\xi}_R = \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l} \end{bmatrix}$$

Angular velocity ω about the vertical

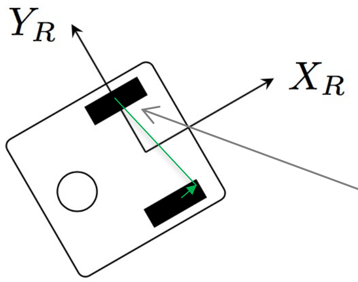


Instantaneous velocities in the robot frame of reference

The motion of the robot in the local robot frame of reference R due to the rotation of the wheels is given by:

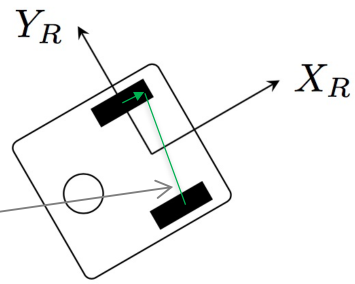
Also see Siegwart et al. (2004, 2011)

$$\dot{\xi}_R = \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l} \end{bmatrix}$$



Contribution to angular velocity about the vertical by the right wheel rotating anticlockwise about the left wheel

Contribution to angular velocity about the vertical by the left wheel rotating clockwise about the right wheel (hence the minus sign)



Odometry-based Position Estimation

Overall approach

- Measure the angular velocities of the left and right wheels
- Compute the instantaneous velocities in the robot frame of reference v, ω
- Compute the displacement and change in orientation (in a given time interval) in the robot frame of reference R

$$v, \omega \downarrow \quad \Delta\xi_R = (\Delta x, \Delta y, \Delta\theta)$$

- Compute the displacement and change in orientation (in a given time interval) in the global frame of reference (inertial frame of reference I)
- Update the position of the robot with respect to its previous position
- Repeat update (e.g. every 100 ms)

Displacement and change in orientation in the robot frame of reference

We need the change in position and orientation of the robot $\Delta\xi = (\Delta x, \Delta y, \Delta\theta)$ rather than its velocity

Unfortunately, we cannot integrate $\dot{\xi}_R = \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l} \end{bmatrix}$ in the general case

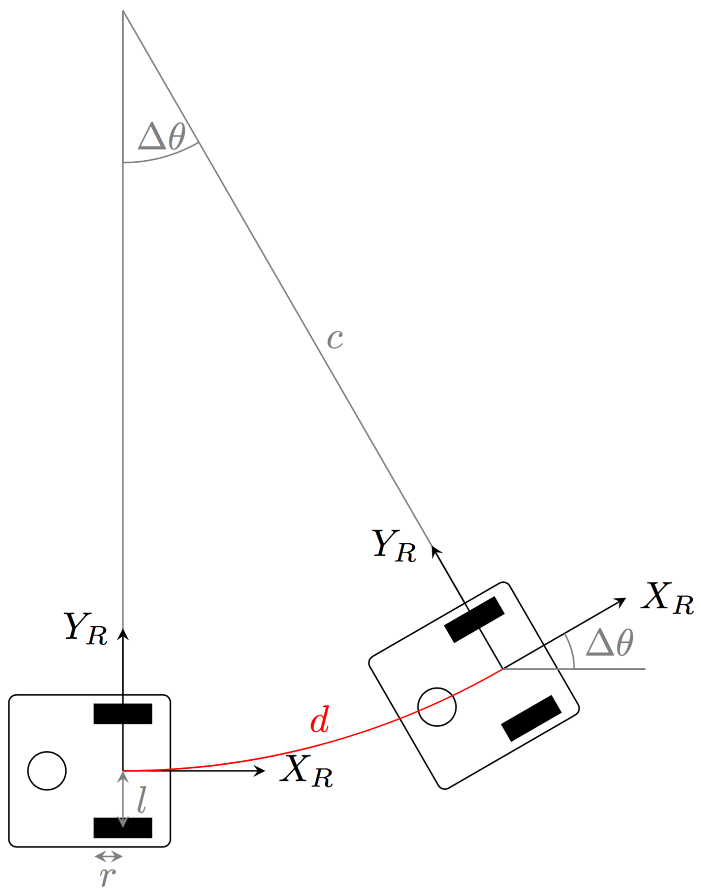
When the robot kinematic constraints (e.g. the rate of change of pose for the two-wheel differential drive robot) is specified with a differential relationship which is not integrable to define the constraint in terms of position variables only, we say the robot is nonholonomic

But we can do it for some important special cases, e.g., constant angular velocity of the wheels

Displacement and change in orientation in the robot frame of reference

If the angular velocities of the wheels are constant,
the path of the robot is constrained to be on a circular arc
with centre known as the instantaneous centre of rotation ICR

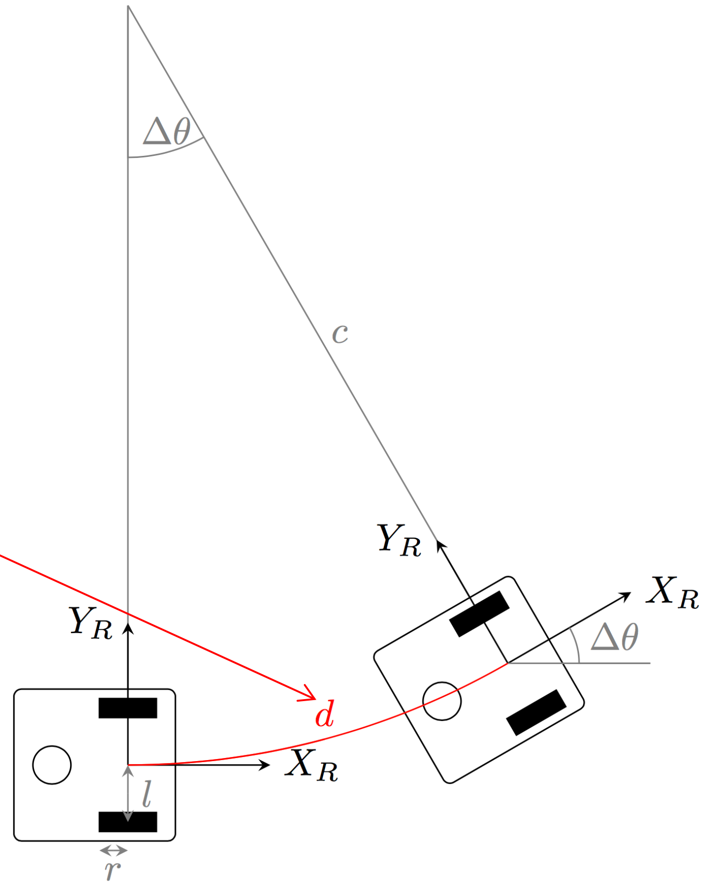
In this case, we can derive an expression for the change in pose
in terms of wheel angular velocity and robot geometry



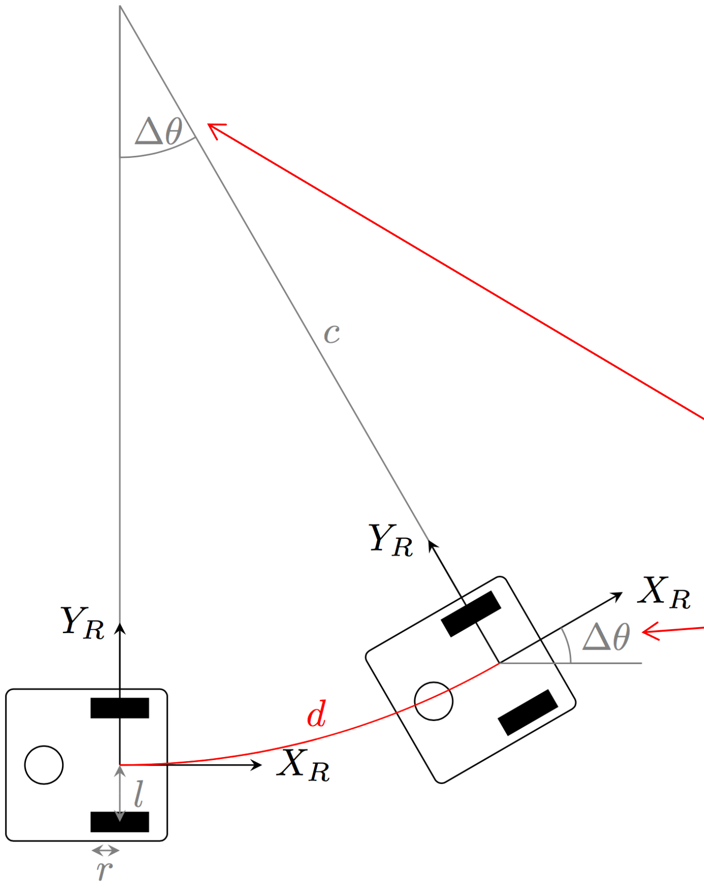
Displacement and change in orientation in the robot frame of reference

The instantaneous velocity in the X_R direction gives the distance d travelled in some unit of time

$$\dot{\xi}_R = \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l} \end{bmatrix}$$



Displacement and change in orientation in the robot frame of reference



$$\dot{\xi}_R = \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l} \end{bmatrix}$$

The angular velocity about the vertical gives the change in orientation (in some unit of time)

Displacement and change in orientation in the robot frame of reference

We have two known values:

$$d = \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2}$$

$$\Delta\theta = \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l}$$

Since the angular velocities of the wheels, $\dot{\phi}_1$ and $\dot{\phi}_2$, are constant, d is the length of an arc on a circle of radius c subtended by the angle $\Delta\theta$

$$\Delta\theta = \frac{d}{c}$$

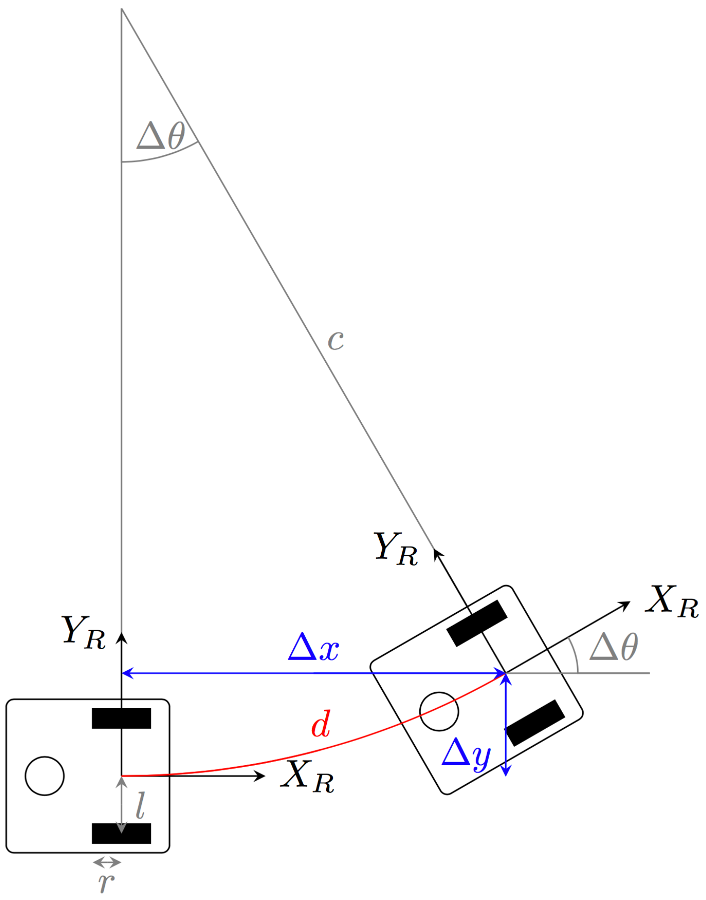
Thus

$$c = \frac{d}{\Delta\theta}$$

and

$$\Delta x = c \sin \Delta\theta$$

$$\Delta y = c - c \cos \Delta\theta$$



Displacement and change in orientation in the robot frame of reference

We have two known values:

$$d = \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2}$$

$$\Delta\theta = \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l}$$

Since the angular velocities of the wheels, $\dot{\phi}_1$ and $\dot{\phi}_2$, are constant, d is the length of an arc on a circle of radius c subtended by the angle $\Delta\theta$

$$\Delta\theta = \frac{d}{c}$$

Thus

$$c = \frac{d}{\Delta\theta}$$

What happens when $\Delta\theta \to 0$?

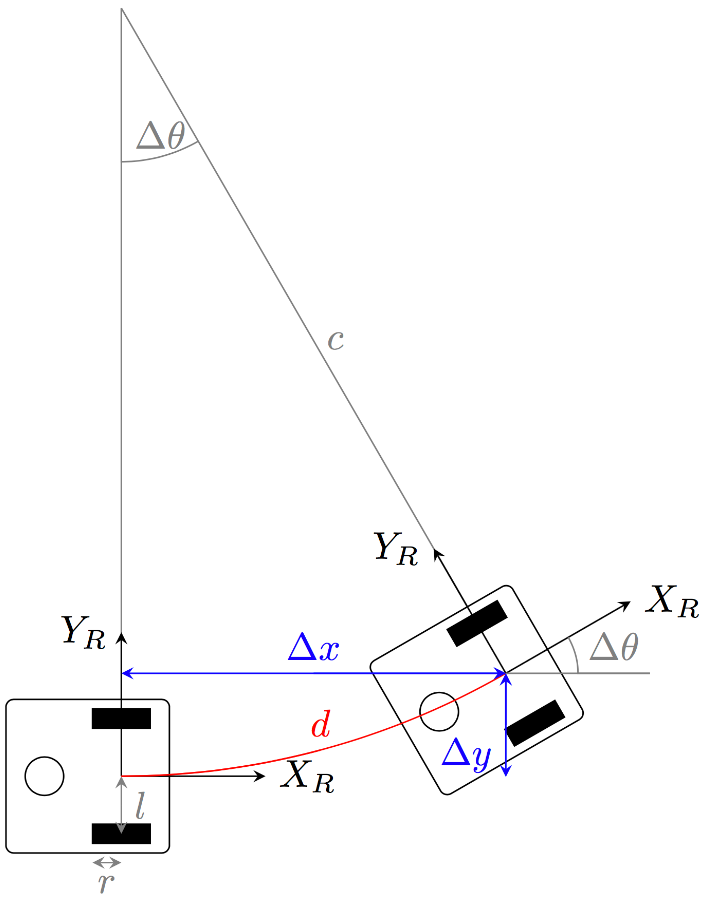
$c \to \infty$
$\sin\Delta\theta \to 0$
$1 - \cos\Delta\theta \to 0$

and

$$\Delta x = c\sin\Delta\theta$$

$$\Delta y = c - c\cos\Delta\theta$$

$\Rightarrow$ large rounding errors when computing Δx and Δy (and division by zero when $\Delta\theta = 0$)



Displacement and change in orientation in the robot frame of reference

We have two known values:

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$$\Delta\theta = \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l}$$

Since the angular velocities of the wheels, $\dot{\phi}_1$ and $\dot{\phi}_2$, are constant, d is the length of an arc on a circle of radius c subtended by the angle $\Delta\theta$

$$\Delta\theta = \frac{d}{c}$$

Thus

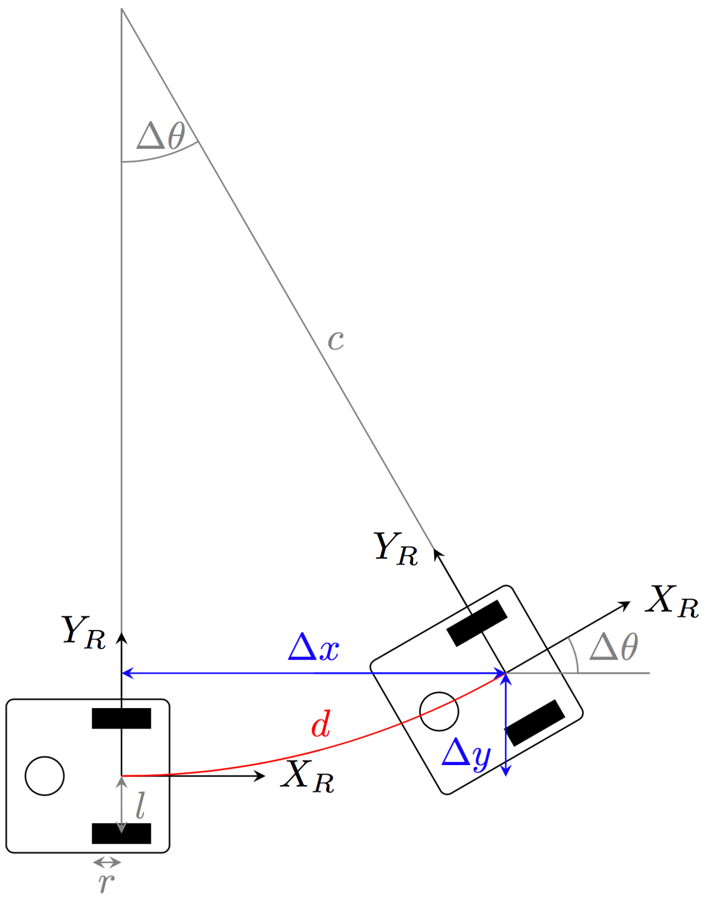
$$c = \frac{d}{\Delta\theta}$$

and

$$\Delta x = c\sin\Delta\theta$$
$$\Delta y = c - c\cos\Delta\theta$$

Solution:
avoid computing c
use computationally better form:

$$\Delta x = d\frac{\sin\Delta\theta}{\Delta\theta}$$
$$\Delta y = d\frac{1-\cos\Delta\theta}{\Delta\theta}$$



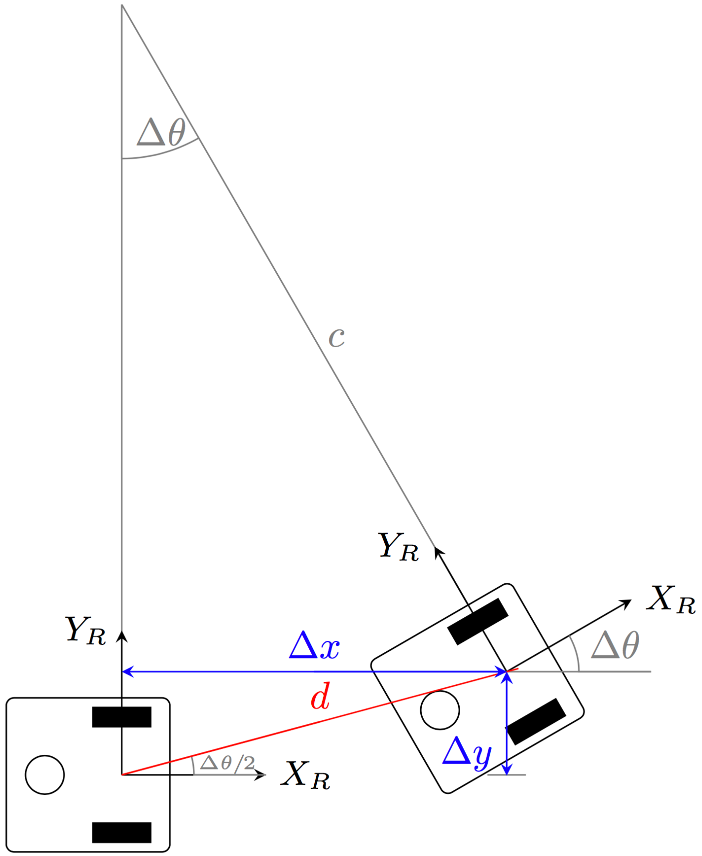
Displacement and change in orientation in the robot frame of reference

For the case where $\Delta\theta = 0$ or, specifically, where $|\Delta\theta| < \text{threshold}$ use either a constant approximation:

$$\begin{aligned} \Delta x &= d \\ \Delta y &= 0 \end{aligned}$$

or a functional approximation:

$$\begin{aligned} \Delta x &= d \cos \frac{\Delta\theta}{2} \\ \Delta y &= d \sin \frac{\Delta\theta}{2} \end{aligned}$$



Displacement and change in orientation in the robot frame of reference

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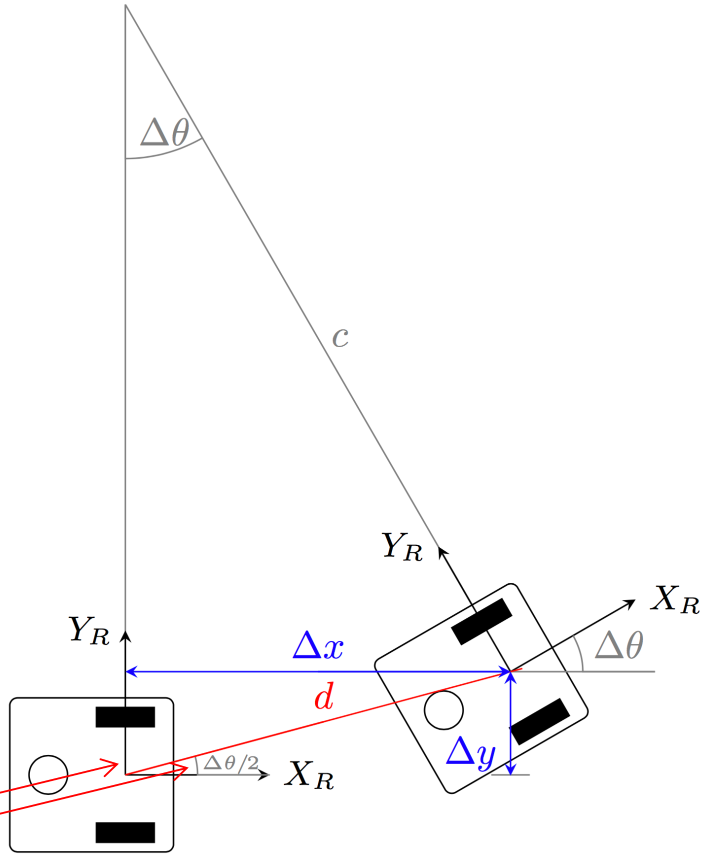


Why $\Delta\theta/2$?

The isosceles triangle formed by the origins of the robot frames of reference and the instantaneous centre of rotation (ICR) has angles $(\pi - \Delta\theta)/2$ at either end of its base

hence the angle here is $\Delta\theta/2$
i.e. $\pi/2 - (\pi - \Delta\theta)/2$

Odometry-based Position Estimation

Overall approach

- Measure the angular velocities of the left and right wheels
- Compute the instantaneous velocities in the robot frame of reference
- Compute the displacement and change in orientation (in a given time interval) in the robot frame of reference R
- Compute the displacement and change in orientation (in a given time interval) in the global frame of reference (inertial frame of reference I)
- Update the position of the robot with respect to its previous position
- Repeat update (e.g. every 100 ms)

$$\Delta\xi_R = (\Delta x, \Delta y, \Delta\theta)$$

$$\downarrow$$

$$\Delta\xi_I = (\Delta x_I, \Delta y_I, \Delta\theta)$$

Displacement and change in orientation in the inertial frame of reference

Change in pose in the robot frame of reference R

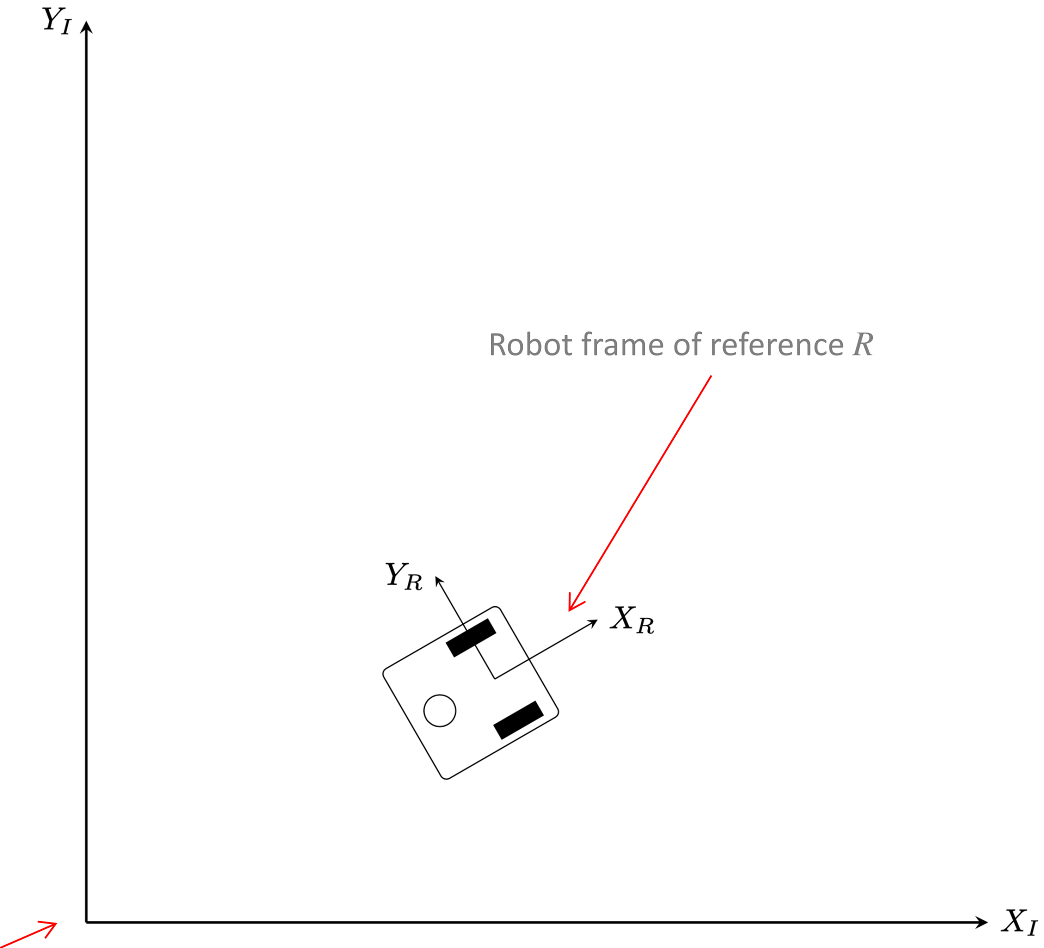
$$\Delta\xi_R = (\Delta x, \Delta y, \Delta\theta)$$

Change in orientation is the same in both frames of reference because the Z axes are aligned

$$\Delta\xi_I = (\Delta x_I, \Delta y_I, \Delta\theta)$$

Change in pose in the inertial frame of reference I

Inertial frame of reference I

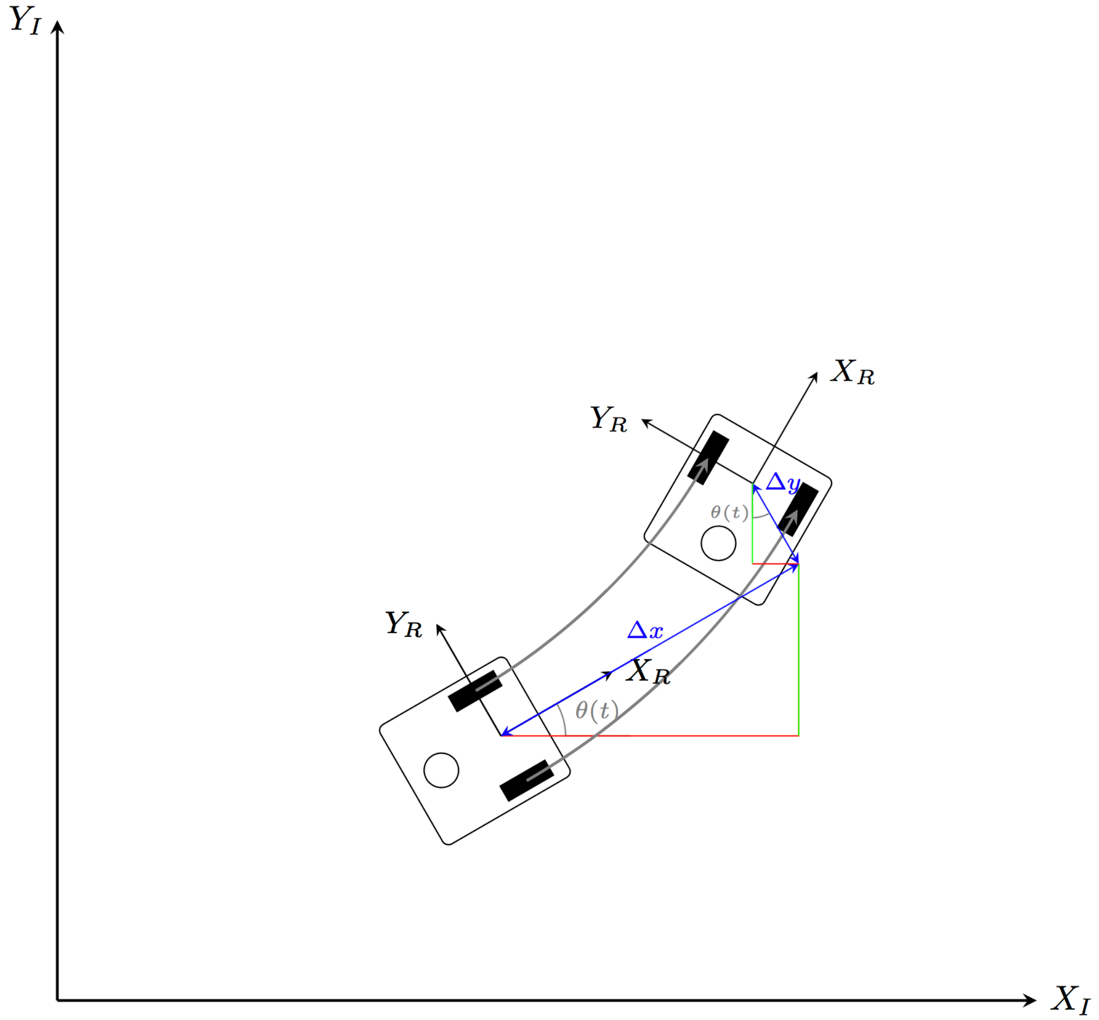


Displacement and change in orientation in the inertial frame of reference

Orientation at time t in the inertial frame of reference

$$\Delta x_I = \Delta x \cos\theta(t) - \Delta y \sin\theta(t)$$

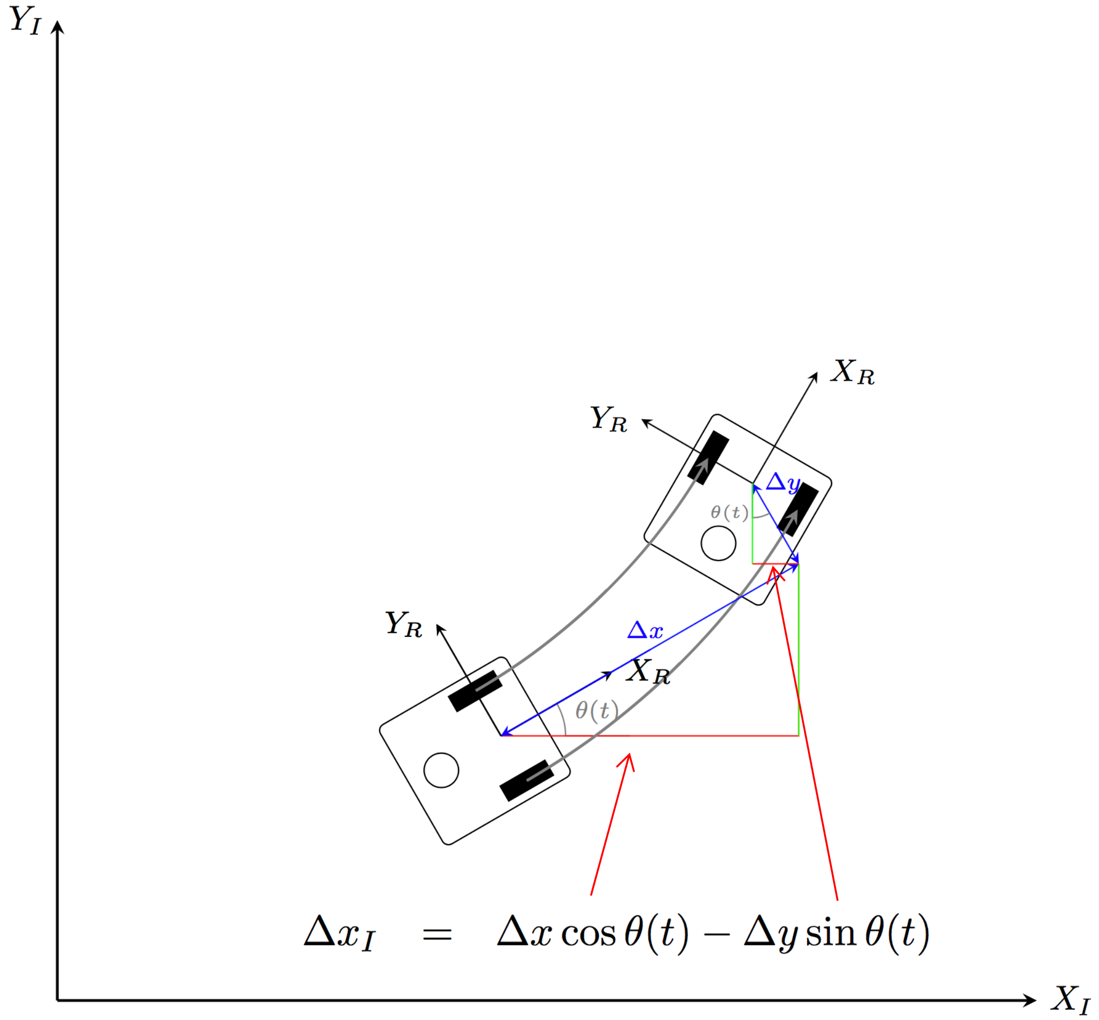
$$\Delta y_I = \Delta x \sin\theta(t) + \Delta y \cos\theta(t)$$



Displacement and change in orientation in the inertial frame of reference

$$\Delta x_I = \Delta x \cos\theta(t) - \Delta y \sin\theta(t)$$

$$\Delta y_I = \Delta x \sin\theta(t) + \Delta y \cos\theta(t)$$



Displacement and change in orientation in the inertial frame of reference

$$\Delta x_I = \Delta x \cos\theta(t) - \Delta y \sin\theta(t)$$

$$\Delta y_I = \Delta x \sin\theta(t) + \Delta y \cos\theta(t)$$