Robotics: Principles and Practice

Module 3: Mobile Robots

Lecture 9: Dijkstra's shortest path algorithm

David Vernon
Carnegie Mellon University Africa

www.vernon.eu

Finding the Shortest Path in a Map

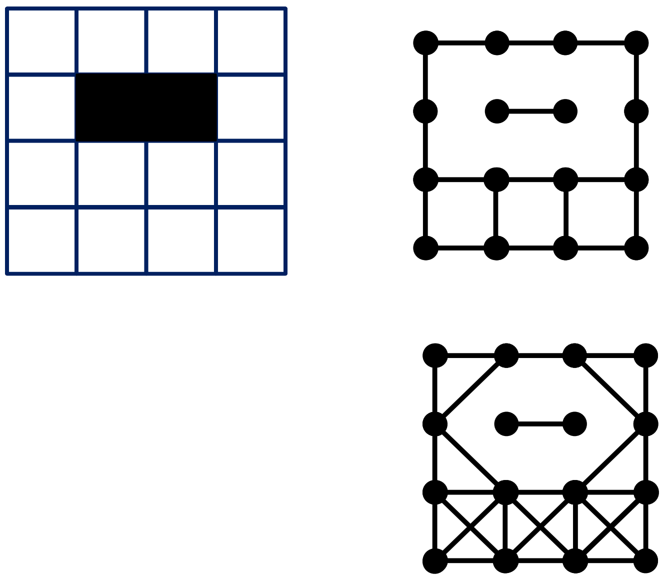
Environment map

– If we represent the map as a graph

  - Free cells are vertices in one or more connected components

  - Obstacle cells are vertices in one or more connected components

    – Not strictly necessary because the robot path is confined to the free space connected component(s)



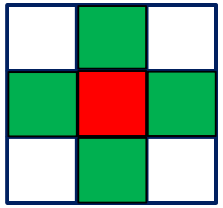
– We can use graph traversal algorithms to find the shortest path connecting a start position and a goal position

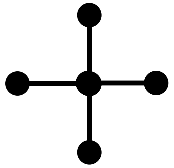
Finding the Shortest Path in a Map

Environment map

Vertices represent free space, i.e. navigable space

What about the edges? There are two possibilities





1. A vertex can be connected to four horizontal neighbour vertices: 4-connectivity

   - All edges represent the same distance, e.g. 1

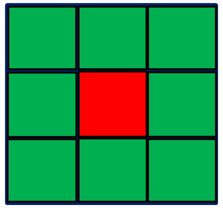
   Use an unweighted graph

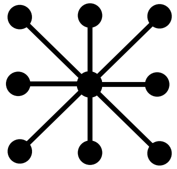
Finding the Shortest Path in a Map

Environment map

2. A vertex can be connected to all eight neighbour vertices: 8-connectivity

- Horizontal edges represent distance of 1
- Diagonal edges represent a distance of $\sqrt{2}$





Need to use a weighted graph:

- weight of 1 for horizontal and vertical edges
- weight $\sqrt{2}$ of for diagonal edges

Finding the Shortest Path in a Map

Environment map

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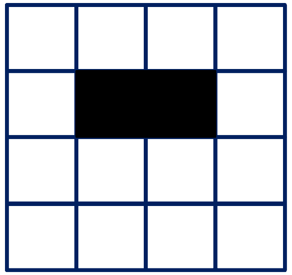

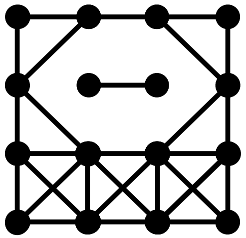

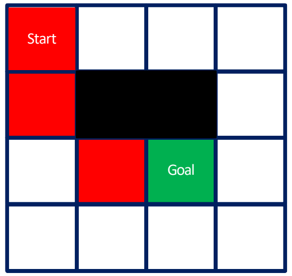

Dijkstra's Shortest Path Algorithm

– Finds shortest path between start and destination vertices

Some implementations find the shortest path between a start vertex and all other vertices, i.e., shortest path spanning tree rooted in the start vertex

– $O(n^2)$ with simple data structures

Dijkstra's Shortest Path Algorithm

- Greedy algorithm
- Repeatedly select the smallest weight edge that will extend the path
  - Begin with some start vertex,
  - Extend the path, one edge at a time
  - Until all vertices are included
- Thus, incrementally construct the shortest path to all vertices

Dijkstra's Shortest Path Algorithm

The principle behind Dijkstra's algorithm is that

if $(s, \ldots, x, \ldots, t)$ is the shortest path from s to t,
then $(s, \ldots, x)$ had better be the shortest path from s to x.

This suggests a dynamic programming-like strategy:

We store the distance from s to all nearby vertices,
and use them to find the shortest path to more distant vertices.

Dijkstra's Shortest Path Algorithm

$known = \{s\}$
for $i = 1$ to n, $dist[i] = \infty$
for each edge (s, v), $dist[v] = d(s, v)$
last=s
while $(last \neq t)$
    select v such that $dist(v) = \min_{unknown} dist(i)$
    for each (v, x), $dist[x] = \min(dist[x], dist[v] + w(v, x))$
    last=v
    $known = known \cup \{v\}$

Select from the unknown vertices

S. Skiena, The Algorithm Design Manual, Springer 2010

Dijkstra's Shortest Path Algorithm

```
ShortestPath-Dijkstra(G, s, t)

path= {s}
for i = 1 to n, dist[i] = ∞
for each edge (s, v), dist[v] = w(s, v)  // initial distances are just the weights
last = s
while (last != t)
    select v_next, the unknown vertex minimizing dist[v]
    for each edge (v_next, x)
        if dist[x] > dist[v_next] + w(v_next, x)
            dist[x] = dist[v_next] + w(v_next, x)
            parent[x] = v_next
    last = v_next
    path = path ∪ {v_next}
```

Dijkstra's Shortest Path Algorithm

ShortestPath-Dijkstra(G, s, t)

The weight of edge (s, v) from vertex s to vertex v

```
path= {s}
for i = 1 to n, dist[i] = ∞
for each edge (s, v), dist[v] = w(s, v)  // initial distances are just the weights
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Dijkstra's Shortest Path Algorithm

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    path = path ∪ {v_next}
```

Extend the path from the vertex with the shortest distance so far

Dijkstra's Shortest Path Algorithm

```
ShortestPath-Dijkstra(G, s, t)

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for i = 1 to n, dist[i] = ∞
for each edge (s, v), dist[v] = w(s, v)  // initial distances are just the weights
last = s
while (last != t)
    select v_next, the unknown vertex minimizing dist[v]
    for each edge (v_next, x)
        if dist[x] > dist[v_next] + w(v_next, x)
            dist[x] = dist[v_next] + w(v_next, x)
            parent[x] = v_next
    last = v_next
    path = path ∪ {v_next}
```

This can be implemented efficiently using a priority queue (implemented as a binary heap)

Dijkstra's Shortest Path Algorithm

ShortestPath-Dijkstra(G, s, t)

path= {s}

for i = 1 to n, dist[i] = ∞

for each edge (s, v), dist[v] = w(s, v) // initial distances are just the weights

last = s

while (last != t)

  select v_{next}, the unknown vertex minimizing dist[v]

  for each edge (v_{next}, x) We now have a new way of reaching x ...

    if dist[x] > dist[v_{next}] + w(v_{next}, x)

if the total distance to x is less than the current distance

      dist[x] = dist[v_{next}] + w(v_{next}, x)

      parent[x] = v_{next}

update the total distance to x

  last = v_{next}

Record the parent of x

  path = path ∪ {v_{next}}

Dijkstra's Shortest Path Algorithm

```
/* Dijkstra's algorithm */

dijkstra(graph *g, int start) {
   int i;                   /* counter                          */
   edgenode *p;             /* temporary pointer                */
   bool intree[MAXV+1];     /* is the vertex in the tree yet?   */
   int distance[MAXV+1];    /* cost of adding to tree           */
   int parent[MAXV+1];      /* parent vertex                    */
   int v;                   /* current vertex to process        */
   int w;                   /* candidate next vertex            */
   int weight;              /* edge weight                      */
   int dist;                /* best current distance from start */

   for (i=1; i<=g->nvertices; i++) {
      intree[i] = FALSE;
      distance[i] = MAXINT;
      parent[i] = -1;
   }

   distance[start] = 0;
   v = start;
```

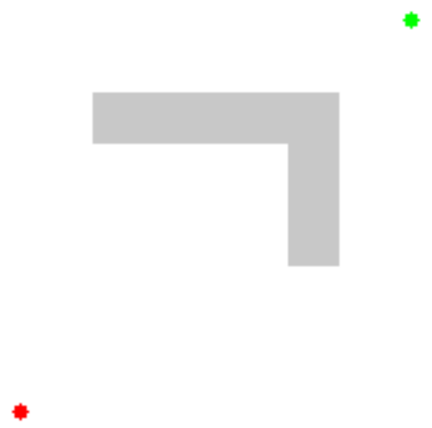
S. Skiena, The Algorithm Design Manual, Springer 2010

Dijkstra's Shortest Path Algorithm

```
    while (intree[v] == FALSE) {
        intree[v] = TRUE;
        p = g->edges[v];
        while (p != NULL) {
            w = p->y;
            weight = p->weight;
            if ((distance[v]+ weight < distance[w])) { // can we improve
                distance[w] = distance[v] + weight;     // on the distance to w?
                parent[w] = v;
            }
            p = p->next;
        }
        v = 1;
        dist = MAXINT;
        for (i=1; i<=g->nvertices; i++)
            if ((intree[i] == FALSE) && (distance[i] < dist)) {
                dist = distance[i];
                v = i;
            }
    }
}
```

S. Skiena, The Algorithm Design Manual, Springer 2010

Dijkstra's Shortest Path Algorithm



"Illustration of Dijkstra's algorithm finding a path from a start node (lower left, red) to a goal node (upper right, green) in a robot motion planning problem. Open nodes represent the "tentative" set (aka set of "unvisited" nodes). Filled nodes are visited ones, with color representing the distance: the greener, the closer. Nodes in all the different directions are explored uniformly, appearing more-or-less as a circular wavefront as Dijkstra's algorithm uses a heuristic identically equal to 0."

https://en.wikipedia.org/wiki/Dijkstra%27s_algorithm

Shortest Paths

Dijkstra's Algorithm

– This implementation finds the shortest path spanning tree, i.e. shortest path between a `start` vertex and all other vertices

– The length of the shortest path from `start` to a given vertex `t` is exactly the value of `distance[t]`

– To find the actual path, follow the `parent` relations from `t` until we hit `start` (or -1 if no such path exists)

– We did this in Breadth-First Search

```
find_path(int start, int end, int parents[])
```